INVENTORY MODELS WITH A MIXTURE OF BACKORDERS AND LOST SALES

Douglas C. Montgomery,
M. S. Bazaraa,
and
Ajit K. Keswani
Georgia Institute of Technology

ABSTRACT

This article presents several single-echelon, single-item, static demand inventory models for situations in which, during the stockout period, a fraction b of the demand is backordered and the remaining fraction $1-b$ is lost forever. Both deterministic and stochastic demand are considered, although the case of stochastic demand is treated heuristically. In each situation, a mathematical model representing the average annual cost of operating the inventory system is developed, and an optimum operating policy derived. At the extremes $b=1$ and $b=0$ the models presented reduce to the usual backorders and lost sales cases, respectively.

INTRODUCTION

The problem of determining economic lot sizes in inventory systems has been treated extensively in recent years. Most of this effort has been concentrated on two general situations regarding the demand process when the system is out of stock; i.e., either all demand during the stockout period is backordered or all demand during the stockout period is lost forever. These two cases result in backorders or lost sales models, respectively. However, in many real inventory systems it is more reasonable to assume that only a fraction, say $b (0 < b < 1)$ of the demand during the stockout period can be backordered, and the remaining fraction $1-b$ is lost. For example, if the inventory item is a spare part some customers whose needs are not critical at the present time can wait for the item to be backordered, while others cannot wait and will fill their demand from another source.

Inventory models which consider a mixture of backorders and lost sales have been suggested and even formulated by several authors [1] [3], but not solved. Moreover, the solution is of interest, as it offers further insight into the nature of inventory processes. It can be shown that making the usual assumptions of all backorders or lost sales when in fact a mixture of the two exists can significantly affect inventory costs.

DETERMINISTIC DEMAND

We consider the single-echelon, single item, static demand case. The following definitions apply throughout this discussion:

D = demand per year

Q = the order quantity

C = unit cost of each item (dollars per unit)

I = inventory carrying cost per year, as a percent of C.

A = fixed ordering cost per inventory cycle (dollars per order)

S = total demand per cycle during the stockout period

π = fixed penalty cost per unit short (dollars per unit)

$\bar{\pi}$ = shortage cost per unit period per backorder (dollars per unit period)

π_0 = profit per unit (dollars per unit)

We assume that the mixture of backorders and lost sales during the stockout period is known and constant. Therefore the total number of demands backordered per inventory cycle is bS, and the total number of demands lost is $(1-b)S$. The inventory geometry for this system is shown in Figure 1. Now the annual revenue received will depend on the length of time for which the system is out of stock, and hence on the operating doctrine. Therefore, both revenue and inventory costs depend upon the operating doctrine, and one would not necessarily obtain the same operating policy from a model which minimizes average annual cost as from a model which maximizes the average annual profit. However, by defining lost sales penalty costs to include lost profits as we do below, either a minimum cost or maximum profit formulation yields identical results.

From the above definitions and a consideration of the inventory geometry we can obtain the average annual cost which is the sum of ordering, carrying, and stockout costs, and which includes lost profit, as

(1)

$$K(Q,S) = \frac{AD}{Q+S(1-b)} + \frac{IC(Q-bS)^2}{2[Q+S(1-b)]} + \frac{\pi SD}{Q+S(1-b)} + \frac{\bar{\pi} bS^2}{2[Q+S(1-b)]} + \frac{\pi_0 SD(1-b)}{Q+S(1-b)}.$$

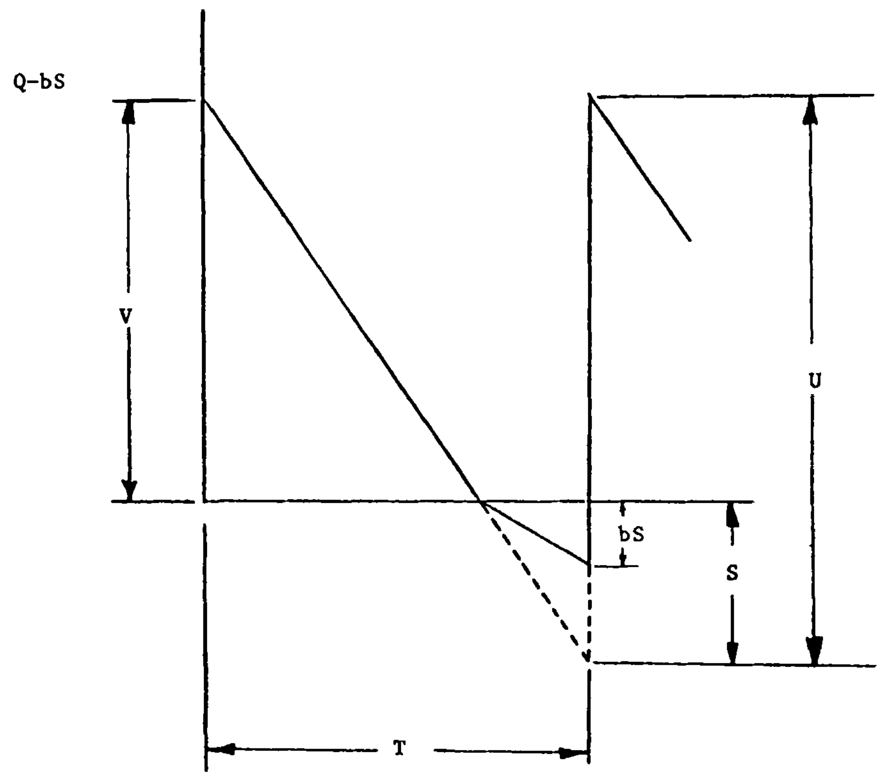

FIGURE 1. A single item deterministic inventory system with the ratio of backorders to demand constant

Note that if $b=1$ or $b=0$, Equation (1) reduces to the usual backorders or lost sales cases, respectively. To find the optimal values of Q and S, say Q^* and S^*, which minimize the average annual cost it is necessary that $\partial K/\partial Q=0$ and $\partial K/\partial S=0$. These conditions result in two simultaneous nonlinear equations in Q and S whose solution is not straightforward. Furthermore, Equation (1) is not convex and we have no guarantee that this will result in a global minimum.

We shall describe a procedure to find the global minimum of K. The following result on nonsingular transformations will be useful:

LEMMA: Let f be a function of $x=(x_1, x_2, \ldots, x_n)$. Define $y=Tx+z$, where $y=(y_1, y_2, \ldots, y_n)$, $z=(z_1, z_2, \ldots, z_n)$, and T is an $(n\times n)$ nonsingular matrix such that $f(x)= f(T^{-1}(y-z))=g(y)$. Then if $g(y^*)\leq g(y)\ \forall\, y\epsilon R^n$, then $f(x^*)=f(T^{-1}(y^*-z))\leq f(x)\ \forall\, x\epsilon R^n$.

PROOF: Since the transformation $y=Tx+z$ is one-to-one, $g(y^*)=f(T^{-1}(y^*-z))\leq g(y)$ $=f(T^{-1}(y-z))\forall\, y\epsilon R^n$. But $x^*=T^{-1}(y^*-z)$ and $x=T^{-1}(y-z)$. Thus $f(x^*)\leq f(x)\forall x\epsilon R^n$.

Consider the following transformation:

$$U=Q+S(1-b) \tag{2}$$

$$V=Q-bS. \tag{3}$$

Here $T=\begin{bmatrix} 1 & 1-b \\ 1 & -b \end{bmatrix}$, and since $|T|=-1$, we have a nonsingular transformation. We see that U is the total demand during the cycle and $V(V\leq U)$ is the onhand inventory at the beginning of the cycle. Using this transformation Equation (1) becomes

$$\begin{aligned}\hat{K}(U,V) &= \frac{AD}{U}+\frac{ICV^2}{2U}+\frac{\pi D(U-V)}{U}+\frac{\bar{\pi} b(U-V)^2}{2U}+\frac{\pi_0 D(1-b)\ (U-V)}{U} \\ &= U^{-1}[\alpha_1+\alpha_2\ (U-V)+\alpha_3\ (U-V)^2+\alpha_4 V^2],\end{aligned} \tag{4}$$

where $\alpha_1=AD$, $\alpha_2=\pi D+\pi_0 D(1-b)$, $\alpha_3=\bar{\pi} b/2$, and $\alpha_4=IC/2$. Since the transformation (2)(3) is nonsingular, if (U^*, V^*) minimize $\hat{K}$ then by the lemma the inverse transformation yields $(Q^*=bU^* + (1-b)V^*, S^*=U^*-V^*)$ which minimize K. However, $\hat{K}$ is not convex so care must be taken to insure that the global minimum is found.

We minimize $\hat{K}$ in two stages. In the first stage we minimize along the ray $V=\beta U$. Once we find the minimum along each ray of this form the best (minimum cost) ray with the property $\beta\leq 1$ is found. For a given value of β, Equation (4) becomes

$$Z(U)=\hat{K}(U, V=\beta U)=\frac{\alpha_1}{U}+\alpha_2(1-\beta)+U[\alpha_3(1-\beta)^2+\alpha_4\beta^2]. \tag{5}$$

The value of U that minimizes (5) must satisfy $\partial Z/\partial U=0$, which results in

$$\begin{aligned}U^* &= \sqrt{\frac{\alpha_1}{\alpha_3(1-\beta)^2+\alpha_4\beta^2}} \\ &= \sqrt{\frac{2AD}{\bar{\pi} b(1-\beta)^2+IC\beta^2}}\end{aligned} \tag{6}$$

We observe that U^* yields an absolute minimum for (5) since $\partial^2 Z/\partial U^2 = 2\alpha_1 U^{-3} \geq 0$ for $U \geq 0$.

Substituting (6) into (5) yields

$$W(\beta) = \alpha_2[1 - \beta + \sqrt{\alpha_5(1-\beta)^2 + \alpha_6\beta^2}], \tag{7}$$

where $\alpha_5 = 4\alpha_1\ \alpha_3/\alpha_2^2$ and $\alpha_6 = 4\alpha_1\ \alpha_4/\alpha_2^2$. Choosing the best ray involves minimizing W with respect to β. Since $\alpha_2 > 0$ minimizing (7) with respect to β for $\beta \leq 1$ is equivalent to minimizing

$$L(\beta) = -\beta + \sqrt{\alpha_5(1-\beta)^2 + \alpha_6\beta^2} \tag{8}$$

for $\beta \leq 1$. The first derivative $L'(\beta)$ is

$$L'(\beta) = -1 + \frac{-\alpha_5(1-\beta) + \alpha_6\beta}{\sqrt{\alpha_5(1-\beta)^2 + \alpha_6\beta^2}}.$$

It can be verified that $L'(\beta)$ is a monotonically increasing function of β, and furthermore

(a) $L'(0) = -1 - \sqrt{\alpha_5} < 0$, and $L'(1) = -1 + \sqrt{\alpha_6}$,

(b) $L'\left(\dfrac{\alpha_5}{\alpha_5 + \alpha_6}\right) = -1$,

(c) $L'(\beta) \to -1 + \sqrt{\alpha_5 + \alpha_6}$ as $\beta \to \infty$, and

$L'(\beta) \to -1 - \sqrt{\alpha_5 + \alpha_6}$ as $\beta \to -\infty$.

The graph of $L'(\beta)$ is shown in Figure 2 (a and b).

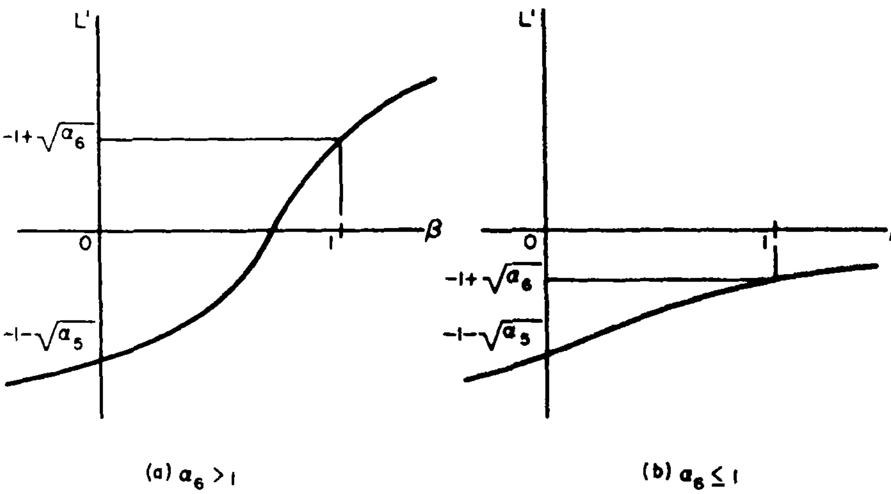

FIGURE 2. Behavior of $L'(\beta)$

We see that if $\alpha_6 \leq 1$ then $L'(1) \leq 0$. Therefore, due to monotonicity, L' is negative over the interval $(-\infty, 1)$. Thus L is decreasing over this interval and the absolute minimum of L is achieved at $\beta^* = 1$. If $\alpha_6 > 1$ then $L'(1) > 0$. Since $L'\left(\frac{\alpha_5}{\alpha_5+\alpha_6}\right) = -1$ then for some $\beta \epsilon \left(\frac{\alpha_5}{\alpha_5+\alpha_6}, 1\right)$ we must have $L'(\beta) = 0$. By monotonicity of L' it is clear that this β corresponds to a global minimum. From $L'(\beta) = 0$ we find the optimum β as

$$\beta^* = \frac{\alpha_5}{\alpha_5+\alpha_6} + \frac{1}{\alpha_5+\alpha_6}\sqrt{\frac{\alpha_5\alpha_6}{\alpha_5+\alpha_6-1}}. \tag{9}$$

This analysis leads to a simple decision procedure to compute the optimum U^*, V^*. Two cases must be considered:

(i) If $\alpha_6 = \frac{2ICA}{D[\pi + \pi_0(1-b)]^2} \leq 1$, then the optimum ray is $\beta^* = 1$. Thus $U^* = V^*$, and from Equation (6)

$$U^* = \sqrt{\frac{2AD}{IC}}.$$

That is, the optimum solution is to allow no backorders or lost sales to occur. The optimum order quantity is $Q^* = U^*$ and $S^* = 0$.

(ii) If $\alpha_6 = \frac{2ICA}{D[\pi + \pi_0(1-b)]^2} > 1$ the optimum β^* is given by Equation (9) and the optimum U^* by Equation (6). The optimum value of V is $V^* = \beta^* U^*$. Q^* and S^* can be computed via the inverse transformation.

It is very easy to demonstrate that inventory systems are sensitive to assumptions regarding the nature of demand during the stockout period. For example, suppose that an item has the following characteristics.

$D = 250$ units/year	$b = 0.7$
$C = \$10$/unit	$\pi = \$0.5$/unit short
$A = \$50$/order	$\bar{\pi} = \$0.1$/unit time per backorder
$I = 0.2$	$\pi_0 = \$2$/unit lost.

Since $\alpha_6 = 0.661 < 1$ case (i) applies. The optimum order quantity is $Q^* = 112$ and the minimum average annual cost is \$224. We see that it is cheaper to order 112 units every cycle and set the reorder point such that no demands occur when the system out is out of stock then to operate with a mixture of backorders and lost sales. To carry this example further, we observe that even though $b \neq 1$ may inventory managers often *assume* that $b = 1$. That is, they assume that all demands during the stockout period can be backordered even though this is not strictly true. Suppose we do this for the above data. From Equations (2–26) and (2–27) on page 45 of Hadley and Whitin [2], which define the optimal Q and S, say Q' and S', for a backorders model we may compute $Q' = 420$ and $S' = 350$. The true average annual cost that would result from using Q' and S' may be found by substituting Q' and S' in Equation (1) as \$274. Thus, failure to use the appropriate model has cost management $\$271 - 224 = \$50$ per year for this single item. The effects of this in a multi-item inventory may be quite substantial.

It is possible to consider many different functional forms for the manner in which the mixture of backorders and lost sales will occur. A constant ratio, as analyzed above, may be satisfactory in many practical situations, but other models may often be easily treated. For example if we assume that the ratio of backorders to demand increases linearly from $p(0 \leq p < 1)$ when the system just goes out of stock to 1 at the end of the cycle just before stock is delivered the average annual cost is identical to (4), except for the term involving time dependent backorder costs, which becomes

$$\frac{\bar{\pi}(4b-1)(U-V)^2}{6U},$$

where $b=(p+1)/2$. We obtain a solution for the optimum system parameters U^* and V^* which is similar to that described above, but with $\alpha_3' = \frac{\bar{\pi}(4b-1)}{6}$ and $b' = (p+1)/3$ substituted for α_3 and b, respectively. Therefore, the analysis for the linearly increasing ratio of backorders to lost sales case is identical to the constant ratio case except for the definition of two constants. We have investigated other situations whose solution is not so simple. However, the identification of the proper functional form may be difficult in practice, and the constant or linear ratio models are probably satisfactory approximations.

STOCHASTIC DEMAND

The models presented in this section are based on the continuous and periodic review models in Chapters 4 and 5 of Hadley and Whitin [2]. We assume that the ratio of backorders to lost sales is constant during the stockout period. Once again, we deal with the single-echelon, single-item, static demand case.

Continuous Review

Consider a reorder point or (Q, r) inventory model. This is often called a continuous review or "transactions reporting" model, because as the stock level reaches r, a quantity Q is ordered and this would require knowledge of the inventory level immediately after each demand. We shall present a heuristic, approximate treatment of the continuous demand case. An analogous development may be given for discrete demand. The following assumptions are required:

1. The unit cost of the item is a constant independent of the order quantity, Q.
2. There is a fixed shortage cost, π, for each unit of demand occurring during the stockout period whether that unit is backordered or lost.
3. There is no time dependent backorder cost, i.e., $\bar{\pi}=0$.
4. The reorder point r, based on the net inventory is positive.
5. There is never more than a single order outstanding.
6. The stockout period during a cycle is small enough to be neglected so that the average number of cycles per year is D/Q, where D is the average annual demand.

If the backorder and lost sales penalty costs are reasonably large (as is often the case), our assumptions and heuristic treatment are reasonable. Further justifications and discussions of these assumptions are given by Hadley and Whitin.

Suppose that the distribution of lead time demand is a continuous density function, say $h(x)$. Now if A is the cost of placing an order, then since the average annual demand is D and since an order

is placed after every Q demands, the average annual cost of placing orders is DA/Q. If the lead time demand is x then the expected demand short at the end of the cycle is given by

$$\eta(r)=\int_r^\infty (x-r)h(x)dx.$$

Thus, the expected number of backorders per cycle are $b\eta(r)$ and the expected demand lost per cycle is $(1-b)\eta(r)$, where b is the ratio of the expected number of backorders to the expected demand short per cycle. The expected net inventory at the beginning of the cycle, assuming that the arrival of an order initiates a cycle is given by

$$Q+r-\mu+(1-b)\eta(r),$$

where μ is the expected lead time demand. Also, the expected net inventory at the end of the cycle is given by

$$r-\mu+(1-b)\eta(r).$$

Therefore, the average annual cost of carrying inventory is

$$IC\left[\frac{Q}{2}+r-\mu\right]+IC(1-b)\eta(r).$$

The annual fixed shortage cost is $D/Q\ \pi\eta(r)$ and the annual lost profit is $D/Q\ \pi_0(1-b)\eta(r)$.

All the components of the average annual variable cost $K(Q, r)$ have been found. $K(Q, r)$ is just he sum of the above components, or

(10) $$K(Q,r)=\frac{AD}{Q}+IC\left(\frac{Q}{2}+r-\mu\right)+\left[IC(1-b)+\frac{\pi D}{Q}+\frac{\pi_0(1-b)D}{Q}\right]\eta(r).$$

Taking the first partial derivatives of (10) with respect to Q and r, equating to zero and solving yields

(11) $$Q=\sqrt{\frac{2D\,[A+\pi\eta(r)+\pi_0\,(1-b)\eta(r)]}{IC}}$$

and

(12) $$H(r)=\frac{QIC}{QIC(1-b)+\pi D+\pi_0 D(1-b)},$$

where $H(r)$ is the complimentary cumulative of $h(x)$, that is

$$H(r)=\int_r^\infty h(x)dx.$$

Note that (12) is meaningless if $H(r)>1$. Equation (10) is convex, and hence the solution Q^*, r^* obtained from (11) and (12) yields an absolute minimum. The iterative procedure described by Hadley and Whitin [2] on page 171 may be used to solve Equations (11) and (12).

The effect of b on the minimum average annual cost, say K_b^*, may be examined. Since $\bar{\pi}=0$, the stockout cost per unit of demand is $\pi+\pi_0(1-b)$. Thus the stockout cost per unit of demand is minimum when $b=1$ (backorders model) and maximum when $b=0$ (lost sales model). Thus, $K_{b=1}^* < K_{b=0}^*$, and for $0<b<1$, $K_{b=1}^* < K_b^* < K_{b=0}^*$. Therefore, if $b \neq 1$, and if management could influence the nature of the demand process when the system is out of stock, they would be willing to pay some penalty $Z \leq K_b^* - K_{b=1}^*$ to insure that $b=1$.

Periodic Review

Now consider a heuristic, approximate treatment of the periodic review model, or the (R, T) model. This system requires the inventory level to be reviewed at each time interval of length T, and at each review time a sufficient quantity is ordered to bring the inventory position up to R. We shall treat the continuous demand case, where D is the average demand rate per year. A similar development may be given for discrete demand. The following assumptions are necessary:

1. The cost J of making a review is independent of the variables R and T.
2. The unit cost of the item is a constant, independent of the quantity ordered.
3. There is a fixed shortage cost π for each unit of demand occurring during the stockout period whether that unit is backordered or lost.
4. There is no time dependent backorder cost.
5. The backorders are incurred in very small quantities so that when an order arrives, it is almost always sufficient to meet any outstanding backorders.
6. When the procurement lead time is a random variable, it is assumed that the orders are received in the same sequence in which they are placed, and furthermore, lead times for different orders can be treated as independent random variables.

Once again, Hadley and Whitin discuss the practicality of these assumptions. In many real inventory systems the heuristic model will suffice.

The annual ordering and review cost is given by L/T, where $L=A+J$. To compute the inventory carrying cost, the period T will be used as the time between the arrival of two successive orders rather than between the placement of two successive orders. Let $h(x; T)dx$ be the probability that x units are demanded in a time T. Hadley and Whitin discuss the nature of $h(x; T)$ when procurement lead time is either a constant or a random variable. Now the expected number of demands short per review period is

$$\gamma(R)=\int_R^\infty (x-R)h(x; T)dx.$$

Thus $b\gamma(R)$ demands are backordered per cycle and $(1-b)\gamma(R)$ demands are lost per cycle. The expected net inventory at the beginning of the period is

$$R-\mu+(1-b)\gamma(R),$$

and the expected net inventory at the end of the period is

$$R-\mu-DT+(1-b)\gamma(R),$$

where μ is the expected value of lead time demand. Since it is assumed that the portion of time the system is out of stock is small relative to T, the average annual cost of holding inventory is approximately

$$IC\left[R-\mu-\frac{DT}{2}+(1-b)\gamma(R)\right].$$

The average annual fixed shortage cost is $\pi/T\gamma(R)$ and the annual expected lost sales penalty is $\pi_0/T(1-b)\gamma(R)$.

The average annual variable cost can now be written as the sum of the above components, or

$$K(R,T)=\frac{L}{T}+IC\left(R-\mu-\frac{DT}{2}\right)+\left[IC(1-b)+\frac{\pi+\pi_0(1-b)}{T}\right]\gamma(R). \tag{13}$$

For a given T, the value of R which minimizes (13) must satisfy $\partial K/\partial R=0$. If $H(R;T)=\int_R^\infty h(x;T)dx$ is the complimentary cumulative of $h(x;T)$, then the optimum R is the solution to

$$H(R;T)=\frac{ICT}{ICT(1-b)+\pi+\pi_0(1-b)}, \tag{14}$$

if $H(R;T)\leq 1$. For a given T the solution to (14) will yield an absolute minimum for (13). Equation (13) may be optimized with respect to R and T by tabulating the minimum $H(R,T)$ with respect to R alone for various values of T, and then choosing the T which produces a minimum value of $H(R,T)$ in this table. The behavior of the minimum average annual cost with respect to b, say K_b^* is identical to that of the continuous review model, and management would always be willing to pay some penalty to insure that $b=1$, as this would reduce the minimum average annual cost which could be achieved.

CONCLUSION

This article has treated inventory processes in which a fraction b of demand during the stockout period is backordered and the remainder is lost forever. In addition to a deterministic model heuristic, approximate treatments of reorder point and periodic review models for stochastic demand have been briefly considered. It is felt that these models are representative of the real nature of inventory systems, and may be useful for the practical solution of inventory problems.

REFERENCES

[1] Fabrycky, W. J. and Banks, J., *Procurement and Inventory Systems, Theory and Analysis* (Reinhold Publishing Corporation, New York, 1967).

[2] Hadley, G. and Whitin, T. M., *Analysis of Inventory Systems* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).

[3] Jelen, F. C. (Editor), *Cost and Optimization Engineering* (McGraw-Hill, Inc., New York, 1970).